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SOME ANALYTIC RESULTS CONCERNING THE MASS SPECTRUM OF YANG–MILLS GAUGE THEORIES ON A TORUS[1]

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When non-abelian gauge fields are enclosed in a box with periodic boundary conditions, the spectrum of the hamiltonian becomes discrete and the energy values can be expanded in a power series of $\lambda = g^{2/3}$ (g: renormalized coupling constant). A method to obtain these expansions is explained and worked out to one-loop order. No numbers for the low-lying levels are given here, but some interesting properties of the mass spectrum already become visible.

1. Introduction and summary

In a recent letter [1], I proposed a universal expansion for the masses of the low-lying stable particles in asymptotically free field theories. The method is based on the observation that the energy spectrum of field theories in a box is discrete and peturbatively computable. The main difficulty then is, to work out the perturbation expansion of the low-lying levels to a sufficiently high order. This problem is here attacked for the case of (pure) SU(n) Yang–Mills gauge theories in $3+1$ dimensions.

Gauge theories on a torus were studied by 't Hooft [5]. He noticed that in addition to the usual symmetries, the hamiltonian commutes with a group of non-trivial transformations related to the centre Z_n of SU(n). They will be described in detail in sect. 2. Here, I only remark that they give rise to a division of the physical Hilbert space into n^3 orthogonal subspaces, the "central sectors"*, which have a property characteristic for super selection sectors: application of *local* operators to the states in a given sector does not lead out of the sector. To go from one sector to another, one must act with global operators such as Wilson loops winding around the torus.

The qualitative results of this paper can be summarized as follows.

(a) All eigenvalues of the hamiltonian can be expanded in a power series of $\lambda = g^{2/3}$, where g is the renormalized coupling constant. At least to one-loop order, the renormalization of the coupling constant needed to make the coefficients finite is the usual one.

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* Subspaces of states with definite electric flux in 't Hooft's terminology.

(b) Because the decomposition of the space of physical states into central sectors stems from a symmetry, one may diagonalise the hamiltonian in each sector separately. It turns out that to all orders of λ, the energy values and multiplicities are exactly the same in every sector. This curious degeneracy is lifted at the non-perturbative level.

(c) Most of the energy levels in a fixed central sector approach some value well above the ground state energy as $g \to 0$. Only a distinguished set of zero momentum states are close to the ground state for small g and are in fact degenerate with it at $g = 0$. The explicit calculations in this paper focus on this special group of energy levels. The result is that the energy differences between these states are exactly equal to the differences of the eigenvalues of an effective anharmonic oscillator hamiltonian H', which is described in more detail below. In particular, the ground state in a given central sector is unique for $g > 0$ and the mass gap in that sector is equal to the difference between the next to lowest and the lowest eigenvalue of H'.

The product of the computational effort made in this paper is the effective hamiltonian H'. This is a differential operator acting on wave functions ψ defined on the space of constant SU(n) gauge potentials c_k ($k = 1, 2, 3$). In other words, with respect to a basis T^a ($a = 1, \ldots, n^2 - 1$) of group generators, we have*

$$c_k = c_k^a T^a , \tag{1}$$

and ψ is just any complex valued square integrable function of the $3(n^2-1)$ real variables c_k^a. A remnant of Gauss' law requires that only those wave functions are physical, which are SU(n) invariant, i.e. wave functions satisfying

$$\psi(\Lambda c \Lambda^{-1}) = \psi(c) , \tag{2}$$

for all $\Lambda \in \mathrm{SU}(n)$. For gauge fields enclosed in an $L \times L \times L$ box, the effective hamiltonian up to one-loop order is then given by

$$H' = \frac{\lambda}{L} \sum_{\nu=0}^{\infty} \lambda^{\nu} H'_{\nu} , \tag{3a}$$

$$H'_0 = \tfrac{1}{2} e_k^a e_k^a + \tfrac{1}{4} (f^{abc} c_k^b c_l^c)(f^{ade} c_k^d c_l^e) , \qquad e_k^a = \frac{1}{i} \frac{\partial}{\partial c_k^a} , \tag{3b}$$

$$H'_1 = a_1 c_k^a c_k^a , \tag{3c}$$

$$H'_2 = 0 , \tag{3d}$$

$$H'_3 = a_2 H'_0 + a_3 s^{abcd} c_k^a c_k^b c_l^c c_l^d + a_4 s^{abcd} c_k^a c_k^b c_k^c c_k^d . \tag{3e}$$

Here, f^{abc} denotes the SU(n) structure constants and s^{abcd} is a totally symmetric

* Repeated indices are always summed over. My conventions on group generators, structure constants etc. are collected in appendix A.

SU(n) invariant tensor (for the precise definition see appendix A). The coefficients a_ν are the values of some one-loop momentum sums. In particular, a_2 is obtained from a logarithmically divergent sum, the divergence being cancelled by the renormalization of the coupling constant as usual. With dimensional regularisation and minimal subtraction [2] (MS scheme) the numbers are

$$a_1 = -\frac{n}{4\pi} 1.89153165\ldots, \tag{4a}$$

$$a_2 = \frac{11n}{9(4\pi)^2}[\ln(\mu L)^2 - 0.409052802\ldots], \tag{4b}$$

$$a_3 = \frac{2}{15(4\pi)^2} - \tfrac{3}{5}a_4, \tag{4c}$$

$$a_4 = -\frac{1}{(4\pi)^2} 0.619331710\ldots. \tag{4d}$$

The lowest order effective hamiltonian H_0' is the hamiltonian one would obtain from the full Yang–Mills action, when the gauge potentials are restricted to depend on time only. Such fields are thus the "slow modes" of the system. The "fast modes" on the other hand are systematically integrated out and their influence on the dynamics of the slow modes is exactly accounted for by perturbations H_ν' ($\nu \geqslant 1$). To generate the perturbation expansion of the individual energy levels it remains to diagonalise H_0'. I do not know whether this can be done analytically, but H_0' has an intriguing algebraic structure and may very well turn out to be of the integrable type. If not one must invoke the "big brother's" help. Some properties of the spectrum of H_0' can however be established without explicit diagonalisation. For example, in sect. 5 of this paper it is shown that there is no continuous spectrum and that the eigenfunctions fall off rapidly for large c_k.

Constant gauge fields can be rotated ($c_k \to R_{kl}c_l$), reflected ($c_k \to -c_k$) and charge conjugated ($c_k \to c_k^*$). The effective hamiltonian is invariant under these operations with the expected restriction that only those rotations are allowed, which do not tilt the box. The last term in eq. (3e) in fact breaks the invariance of H' under the full O(3) down to only the cubic group. The symmetries of H' discussed here are images of the corresponding symmetries of the full hamiltonian. In particular, if $\psi(c)$ is an eigenstate of H' with definite J^{PC}, the corresponding eigenstate of the full hamiltonian in the zero "electric flux" sector has the same J^{PC}.

In order to make this article readable, the more technical proofs and derivations are deferred to appendices. Sect. 2 introduces the reader to gauge fields on a torus, torons (=gauge fields with no magnetic energy) and central conjugations. In sect. 3 it is shown that most torons are quantum mechanically unstable and that the perturbation expansion consequently amounts to an expansion about the classical vacuum configuration as one might have expected naively. The analysis of the low

order expressions then reveals (sects. 4, 5) that the energy spectrum has the general form described above and that the computation of higher order terms is a case of degenerate perturbation theory. In sect. 6 a useful formulation of degenerate perturbation theory due to Bloch [3] is reviewed and then applied to obtain the one-loop effective hamiltonian (sect. 7). The final sect. 8 contains a few concluding remarks.

Some of the results of this paper have been anticipated in a lecture by Bjorken about the "femtouniverse" [4]. This is a hypothetical world about 1 fm wide with tiny physicists, the "femtophysicists", who are studying the strong interactions among quarks and gluons. In what follows, the reader is thus invited to assume a femtophysicist's point of view.

2. Yang–Mills gauge fields on a torus

Let T^3 denote the 3-dimensional torus $S^1 \times S^1 \times S^1$, where each factor S^1 has a volume (circumference) equal to L. A scalar field on T^3 can thus be identified with a field $\phi(x_1, x_2, x_3)$ on $\mathbb{R}^3$, which is periodic in all coordinates x_k with period L. For an SU(n) gauge field on T^3, the situation is initially more complicated, because periodicity is required only modulo gauge transformations. However, a detailed analysis shows that any such gauge field is gauge equivalent to a periodic vector field*. Without loss, gauge fields on T^3 can therefore be written as

$$A_k(x) = A_k^a(x) T^a, \qquad k = 1, 2, 3, \tag{5a}$$

$$A_k(x + L\hat{\imath}) = A_k(x), \tag{5b}$$

where "$\hat{\imath}$" denotes the unit vector in the ith direction. Two gauge fields A_k and $\tilde{A}_k$ are called gauge equivalent, if there is an SU(n) valued *periodic* function $\Lambda(x)$ such that

$$\tilde{A}_k(x) = \Lambda(x) A_k(x) \Lambda(x)^{-1} + \Lambda(x) \partial_k \Lambda(x)^{-1}. \tag{6}$$

In the $A_0 = 0$ gauge, the hamiltonian H acts on wave functionals $\psi[A]$, where the variable A runs over all gauge potentials on T^3 as described above. Gauss' law requires that ψ is invariant under (time independent) gauge transformations, i.e.

$$\psi[\tilde{A}] = \psi[A], \tag{7}$$

for all gauge equivalent potentials A and $\tilde{A}$. The color electric field $E_k^a(x)$, the color magnetic field $B_k^a(x)$ and the hamiltonian H are then given by

$$E_k^a(x) = \frac{1}{i} \frac{\delta}{\delta A_k^a(x)}, \tag{8}$$

* In mathematically precise terms, a gauge field is a connection in an SU(n) principal bundle over T^3. Every such bundle is trivial, i.e. of the form $SU(n) \times T^3$. This means that gauge fields can be identified with Lie algebra valued vector fields on T^3. Twisted gauge fields [5] are connections in non-trivial $SU(n)/Z_n$ principal bundles. I do not consider this possibility here.

$$B_k^a(x) = \tfrac{1}{2}\varepsilon_{klj}(\partial_l A_j^a(x) - \partial_j A_l^a(x) + f^{abc}A_l^b(x)A_j^c(x)), \tag{9}$$

$$H = \int_0^L \mathrm{d}^3x \left\{ \tfrac{1}{2}g_0^2 E_k^a(x)E_k^a(x) + \frac{1}{2g_0^2}B_k^a(x)B_k^a(x) \right\}. \tag{10}$$

Here, g_0 denotes the bare coupling constant and an ultra-violet regularisation is implicitly assumed. For the computation of the one-loop effective hamiltonian dimensional regularisation will be used, i.e. the theory is formulated on a d-dimensional torus T^d and the bare coupling constant is expanded in powers of the renormalized coupling g^2 according to

$$g_0^2 = \mu^{2\varepsilon}\left\{ g^2 - \frac{11n}{3\varepsilon}\frac{g^4}{(4\pi)^2} + \mathrm{O}(g^6) \right\}, \qquad \mathrm{d} = 3 - 2\varepsilon\,. \tag{11}$$

In order to keep the magnetic energy bounded, the wave functionals $\psi[A]$ of the low-lying states have to be supported essentially around the potentials A_k with $B_k = 0$ in the small coupling limit. A detailed description of the solutions of $B_k = 0$ is therefore needed. Following ref. [6], they will be called "torons". At first sight, one might think that torons are simply pure gauge configurations. In fact, $B_k = 0$ implies

$$A_k(x) = \Lambda(x)\partial_k\Lambda(x)^{-1},$$

in every simply connected patch of T^3. But T^3 is not simply connected and Wilson loops that wind around the world can assume non-trivial values even in $B_k = 0$ everywhere. It is not difficult to find the general toron solution. A complete description is given by the following two statements:

(a) For any set of angles

$$\varphi_k^\alpha \in \mathbb{R}, \qquad k = 1, 2, 3, \qquad \alpha = 1, \ldots, n, \qquad \sum_\alpha \varphi_k^\alpha = 0, \tag{12}$$

define the (abelian, constant) gauge potential

$$A_k[\varphi] = \frac{1}{iL}\begin{pmatrix} \varphi_k^1 & & 0 \\ & \ddots & \\ 0 & & \varphi_k^n \end{pmatrix}. \tag{13}$$

Then, every toron solution is gauge equivalent to a gauge potential of the form $A_k[\varphi]$, and every $A_k[\varphi]$ is a toron solution.

(b) Two fields $A_k[\varphi]$ and $A_k[\tilde{\varphi}]$ are gauge equivalent if and only if

$$\tilde{\varphi}_k^\alpha = \varphi_k^{\sigma(\alpha)} \pmod{2\pi}, \tag{14}$$

for some permutation σ (and all k, α).

In other words, the gauge equivalence classes of torons can be labelled by the sets of angles φ_k^α, where any two sets related by eq. (14) are to be identified. This

makes up a compact manifold *with boundary*, the boundary points being characterized by

$$\varphi_k^{\alpha} = \varphi_k^{\sigma(\alpha)} (\mathrm{mod}\, 2\pi)\,, \tag{15}$$

for some non-trivial permutation σ. The fact that the toron manifold is not open causes considerable difficulties in perturbation theory (see ref. [6]). For the computation of energy levels one is however able to overcome these dfficulties (sect. 3).

The final topic in this section are the "central conjugations". These are the extra symmetries alluded to in the introduction. They are defined as follows. Let W be the diagonal $n \times n$ matrix with

$$W_{11} = W_{22} = \cdots = W_{(n-1)(n-1)} = i/n\,,$$

$$W_{nn} = i(1-n)/n\,.$$

W is an element of the Lie algebra of SU(n). For any triplet

$$z_k = \mathrm{e}^{i(2\pi/n)\nu_k}\,, \qquad k = 1, 2, 3\,, \qquad \nu_k \in \{0, 1, \ldots, n-1\}\,, \tag{16}$$

of elements of Z_n define

$$\Lambda_z(x) = \exp\left(\frac{2\pi}{L} \nu_k x_k W\right).$$

This is an SU(n) valued function on $\mathbb{R}^3$, which is quasiperiodic

$$\Lambda_z(x + L\hat{k}) = z_k \Lambda_z(x)\,. \tag{17}$$

The central conjugate $C_z A$ of an arbitrary gauge field A on T^3 is then defined by

$$C_z A_k(x) = \Lambda_z(x) A_k(x) \Lambda_z(x)^{-1} + \Lambda_z(x) \partial_k \Lambda_z(x)^{-1}\,. \tag{18}$$

This transformation has the following properties.

(a) Because the phases z_k commute with matrices, $C_z A$ is periodic and hence a gauge field on T^3.

(b) Locally, C_z is just a gauge transformation, but globally it is not: Wilson loops that wind around T^3 in general change their phase by a multiple of $2\pi/n$.

(c) C_z maps gauge equivalent fields onto gauge equivalent fields and can therefore be considered a mapping of gauge equivalence classes.

In quantum theory, central conjugations are unitarily represented by operators U_z:

$$(U_z \psi)[A] = \psi[C_z^{-1} A]\,. \tag{19}$$

On the space of gauge invariant wave functionals ψ, we have

$$U_z \cdot U_w = U_{z \cdot w}, \qquad z \cdot w = (z_1 w_1, z_2 w_2, z_3 w_3)\,, \tag{20}$$

i.e. the central conjugations make up a group isomorphic to $Z_n \times Z_n \times Z_n$. Furthermore, because of property (b) above, the hamiltonian H commutes with the operators U_z. Central conjugations are therefore genuine symmetries of the system.

The division of the physical Hilbert space into central sectors now comes about as follows. Because of the multiplication law (20), the operators U_z commute and can be simultaneously diagonalised. Furthermore, the group structure requires that the eigenvalues are characters of $\mathbf{Z}_n \times \mathbf{Z}_n \times \mathbf{Z}_n$:

$$U_z\psi = (z_1)^{e_1}(z_2)^{e_2}(z_3)^{e_3}\psi\,, \qquad e_k \in \{0, 1, \ldots, n-1\}\,. \tag{21}$$

There are thus n^3 different choices for the quantum numbers e_k ("electric fluxes" according to 't Hooft [5]) and the central sectors are just the corresponding eigenspaces. Their superselection character (cf. sect. 1) is a consequence of property (b) above, which implies that U_z commutes with all local gauge invariant operators (and arbitrary linear combinations of these).

3. Quantum mechanical instability of torons

Let us start with some heuristic considerations. The hamiltonian H (eq. (10)) has the Schrödinger form with the color electric part playing the rôle of the kinetic energy and the color magnetic part being the potential energy. The torons minimize the potential energy. In directions orthogonal to the toron manifold, the potential energy increases, i.e. the torons are at the bottom of a potential valley, the "toron valley". As $g_0 \to 0$, the wave functionals $\psi[A]$ with small energy are squeezed into the toron valley. Now the following observations are crucial.

(a) The kinetic energy for motions along the toron manifold is of order g_0^2. For $g_0 \to 0$, there is therefore no energetic motivation for the wave functionals $\psi[A]$ to spread along the toron valley.

(b) As will be worked out below, the toron valley is not everywhere equally wide. By moving to places where the valley is widest, the wave functionals $\psi[A]$ can gain energy of order $(g_0)^0$.

Taken together, (a) and (b) imply that the wave functionals of the low-lying states are not spread along the whole toron valley, but rather are supported on perturbative neighborhoods of those few torons, around which the toron valley is widest. What will be shown in this section is that these special torons (i.e. the "quantum mechanically stable" ones) are exactly the classical vacuum $A_k = 0$ and its central conjugates.

We now proceed to make the above argumentation mathematically precise. First, the gauge fields A_k in the neighborhood of the toron manifold are parametrized as follows:

$$A_k(x) = \Lambda(x)\{A_k[\varphi] + g_0 q_k(x)\}\Lambda(x)^{-1} + \Lambda(x)\partial_k\Lambda(x)^{-1}\,. \tag{22}$$

Here, q_k denotes a fluctuation field orthogonal to the toron manifold and the gauge orbit of $A_k[\varphi]$, i.e.

$$\int \mathrm{d}^3x\ (q_k(x))_{\alpha\beta} = 0, \qquad \text{for} \qquad \alpha = \beta\,, \tag{23a}$$

$$D_k q_k = 0\,, \qquad D_k = \partial_k + \mathrm{Ad}\, A_k[\varphi]\,, \tag{23b}$$

(Ad A_k is defined in appendix A). Because the wave functionals considered are gauge invariant, we have

$$\psi[A] = \hat{\psi}[\varphi, q]\,, \tag{24a}$$

$$H\psi[A] = \hat{H}\hat{\psi}[\varphi, q]\,, \tag{24b}$$

for some "reduced" wave functional $\hat{\psi}$ and hamiltonian $\hat{H}$. To lowest order in g_0 one finds

$$\hat{H} = \int_0^L \mathrm{d}^3x\, \{\tfrac{1}{2} p_k^a(x) p_k^a(x) + \tfrac{1}{2} q_k^a(x) (\Omega[\varphi] q)_k^a(x)\}\,, \tag{25}$$

where p_k is the momentum canonically conjugate to q_k and

$$(\Omega[\varphi] q)_k = -D_l D_l q_k + D_l D_k q_l\,. \tag{26}$$

A remarkable fact about eq. (25) is that there is no kinetic part for the variables φ_k^α (cf. observation (a) above). $\hat{H}$ can therefore be diagonalised at any fixed set of angles φ_k^α. In particular, the lowest energy value at φ is

$$\mathscr{E}_0[\varphi] = \tfrac{1}{2}\,\mathrm{Tr}\,(\Omega[\varphi])^{1/2}\,. \tag{27}$$

The ground state of $\hat{H}$ is thus obtained by minimizing $\mathscr{E}_0[\varphi]$ (cf. observation (b) above). In this way, the quantum mechanically stable torons are singled out. Note that to any order in perturbation theory, the ground state and the other low-lying states are confined to a small neighborhood of the stable torons, because the kinetic energy part for the variables φ_k^α is of order g_0^2 and cannot compete with the rise of $\mathscr{E}_0[\varphi]$ away from its minima.

It remains to compute $\mathscr{E}_0[\varphi]$. This is possible, because the covariant derivatives D_k commute with each other and with $\Omega[\varphi]$. It is then not difficult to determine the eigenvalues of $\Omega[\varphi]$ and to compute the frequency sum (27) (appendix C). The outcome is

$$\mathscr{E}_0[\varphi] = \mathscr{E}_0[0] + \frac{1}{\pi^2 L} \sum_{\alpha \neq \beta} \sum_{\nu \neq 0} \frac{1}{(\nu^2)^2} (1 - \cos \nu \cdot (\varphi^\alpha - \varphi^\beta))\,. \tag{28}$$

Here, ν runs over all vectors (ν_1, ν_2, ν_3) of integers and $\varphi^\alpha = (\varphi_1^\alpha, \varphi_2^\alpha, \varphi_3^\alpha)$. It follows immediately that $\mathscr{E}_0[\varphi]$ assumes the minimal value $\mathscr{E}_0[0]$ if and only if

$$\varphi_k^\alpha = \varphi_k^\beta (\mathrm{mod}\, 2\pi)\,, \qquad \text{for all} \qquad k = 1, 2, 3;\ \alpha, \beta = 1, \ldots, n\,. \tag{29}$$

Because $\sum_\alpha \varphi_k^\alpha = 0$, this condition amounts to

$$\varphi_k^\alpha = \frac{2\pi}{n} \nu_k (\mathrm{mod}\, 2\pi)\,, \qquad \nu_k \in \{0, 1, \ldots, n-1\}\,, \tag{30}$$

i.e. the stable torons are characterised by a triplet of integers ν_k (mod n). Their

total number is n^3, which is exactly equal to the number of central conjugates of the classical vacuum $A_k = 0$. From this observation (or by inspection) one arrives at the conclusion that all torons are quantum mechanically unstable except the classical vacuum and its central conjugates*.

4. Leading order perturbation theory

The result of sect. 3 implies that to generate the perturbative expansion of the eigenfunctionals and eigenvalues of H, one must expand about the classical vacuum $A_k = 0$ or any of its central conjugates. Each expansion yields eigenfunctionals, which are supported essentially on a small neighborhood of the field one is expanding about. Eigenfunctionals belonging to different expansion points are mapped onto each other by the central conjugations U_z (eq. (19)). Thus, if ψ is an eigenfunctional of H, which is obtained by perturbing around $A_k = 0$, the conjugate states $U_z\psi$ are degenerate with ψ and can be linearly combined in such a way as to get eigenfunctionals lying in definite central sectors. It follows that to all orders of perturbation theory, the spectrum of H is the same in every central sector. Exchange effects via the toron valleys lift this degeneracy at the non-perturbative level and the perturbative ground state, which is invariant under central conjugations, is promoted to the unique true ground state. To compute the spectrum of H in any given central sector perturbatively, it is however sufficient to work out the expansion about $A_k = 0$**.

Before the expansion about the classical vacuum can be effected, the gauge degrees of freedom must be eliminated. The gauge fixing procedure in the hamiltonian formulation has been worked out in great detail by Christ and Lee [7] (earlier references are given in that paper). I shall therefore be rather sketchy on this point. To be prepared for dimensional regularization, the base space is taken to be a d-dimensional torus T^d from now on. Thus, let Λ_1^{T} denote the space of gauge fields $A_k(x)$ on T^d, which are transverse:

$$\partial_k A_k = 0\,, \tag{31}$$

(we are heading for the Coulomb gauge). For a gauge invariant wave functional $\psi[A]$ it is sufficient to know its values along Λ_1^{T}, because gauge fields off Λ_1^{T} can be made transverse by a gauge transformation. The reduced wave functional

$$\hat{\psi} = \psi|_{\Lambda_1^{\mathrm{T}}}\,, \tag{32}$$

* As a check, I computed $\mathscr{E}_0[\varphi]$ in lattice gauge theories. In the continuum limit (lattice spacing $a \to 0$) the result (28) was reproduced. For any finite a, $\mathscr{E}_0[\varphi]$ differs from the continuum expression, but the torons that minimize $\mathscr{E}_0[\varphi]$ are the same.

** The reader may have noticed the analogy to the double well anharmonic oscillator, where one can expand about either of the two minima of the potential. Because of the potential hump between the expansion points, the wave functions in the two wells do not communicate on a perturbative level. In the gauge theory, on the other hand, a perturbative exchange does not take place, because the connecting toron valley is too narrow.

therefore contains all information on ψ. Since $H\psi$ is also gauge invariant, there must exist a linear operator $\hat{H}$ acting on reduced wave functionals such that

$$\hat{H}\hat{\psi} = \widehat{H\psi}\,. \tag{33}$$

$\hat{H}$ is the Hamilton operator in the Coulomb gauge. By construction, the eigenvalues of $\hat{H}$ are those of the full hamiltonian H (in the sector of gauge invariant states) and the corresponding eigenfunctionals are related by eq. (32). Note that the Coulomb gauge condition (31) tolerates constant gauge rotations and that physical reduced wave functionals must therefore be invariant under these transformations.

Let π_k^a be the transverse part of the color electric field (eq. (8)):

$$\pi_k^a(x) = E_k^a(x) - \left(\partial_k \frac{1}{\Delta}\partial_j E_j^a\right)(x)\,, \qquad \Delta = \partial_j\partial_j\,. \tag{34}$$

π_k^a makes sense as an operator acting on reduced wave functionals. The reduced hamiltonian is then given by [7]*

$$\hat{H} = \tfrac{1}{2}g_0^2 \int_0^L \mathrm{d}^d x\, \mathrm{d}^d y\, \rho^{-1/2}\pi_k^a(x)\rho^{1/2}K(x,y)_{kj}^{ab}\rho^{1/2}\pi_j^b(y)\rho^{-1/2}$$

$$+\frac{1}{2g_0^2}\int_0^L \mathrm{d}^d x\, B_k^a(x)B_k^a(x)\,. \tag{35}$$

Here, ρ is a kind of Faddeev–Popov determinant,

$$\rho = \det{}'(-\partial_k D_k)\,, \qquad D_k = \partial_k + \mathrm{Ad}\, A_k\,, \tag{36}$$

and K incorporates the non-abelian Coulomb Green function:

$$K(x,y)_{kj}^{ab} = \delta^{ab}\delta_{kj}\delta(x-y) + \left(\mathrm{Ad}\, A_k \frac{1}{\partial_l D_l}\Delta\frac{1}{\partial_m D_m}\mathrm{Ad}\, A_j\right)^{ab}(x,y)\,, \tag{37}$$

(in eqs. (36) and (37) the trivial constant zero modes of $\partial_k D_k$ are to be omitted).

We are now well prepared to expand about $A_k = 0$. To this end we substitute

$$A_k^a(x) = g_0^{2/3}L^{-d/3}c_k^a + g_0 q_k^a(x)\,, \tag{38a}$$

$$c_k^a\text{: independent of } x\,, \qquad \int_0^L \mathrm{d}^d x\, q_k^a(x) = 0\,, \tag{38b}$$

* The scalar product of two gauge invariant wave functionals $\psi[A]$ and $\chi[A]$ is

$$(\psi,\chi) = \int \mathscr{D}[A]\psi[A]^*\chi[A] = \int_{A_1^T} \mathscr{D}[A]\rho[A]\hat{\psi}[A]^*\hat{\chi}[A]\,,$$

$\hat{H}$ as defined by eq. (33) is hermitian relative to this scalar product. For perturbation theory, the field dependent measure factor $\rho[A]$ is disturbing. However, a similarity transformation

$$\hat{\psi} \to \rho^{1/2}\hat{\psi}\,, \qquad \hat{H} \to \rho^{1/2}\hat{H}\rho^{-1/2}\,,$$

removes the obstacle. Eq. (35) displays the transformed $\hat{H}$.

and expand for $g_0 \to 0$. The reason for the separation and unusual scaling of the constant modes c_k^a will become clear below. The canonical momenta π_k^a are decomposed and scaled similarly:

$$\pi_k^a(x) = g_0^{-2/3} L^{-2d/3} e_k^a + g_0^{-1} p_k^a(x), \tag{39a}$$

$$e_k^a = \frac{1}{i}\frac{\partial}{\partial c_k^a}, \qquad \int_0^L \mathrm{d}^d x\, p_k^a(x) = 0. \tag{39b}$$

When eqs. (38) and (39) are inserted into eq. (35), the following expansion results:

$$\hat{H} = H_0 + g_0^{2/3} H_1, \tag{40a}$$

$$H_1 = \sum_{\nu=0}^{\infty} g_0^{\nu/3} H_1^{(\nu)}, \tag{40b}$$

$$H_0 = \tfrac{1}{2}\int_0^L \mathrm{d}^d x\, \{p_j^a p_j^a + \partial_l q_j^a \partial_l q_j^a\}. \tag{41}$$

The lowest order hamiltonian H_0 is harmonic and can be diagonalised by the Fourier transformation:

$$q_j^a(x) = \frac{1}{L^d}\sum_{k\neq 0} \{\mathrm{e}^{ik\cdot x} b_j^a(k) + \mathrm{e}^{-ik\cdot x} b_j^a(k)^+\}, \tag{42a}$$

$$p_j^a(x) = \frac{1}{L^d}\sum_{k\neq 0} (-i)|k|\{\mathrm{e}^{ik\cdot x} b_j^a(k) - \mathrm{e}^{-ik\cdot x} b_j^a(k)^+\}, \tag{42b}$$

$$k = (k_1, \ldots, k_d), \qquad k_l = \frac{2\pi}{L}\nu_l, \qquad (\nu_l \in \mathbb{Z}). \tag{42c}$$

Note that b and b^+ are not defined for $k = 0$. The most important properties of these operators are

$$k_j b_j^a(k) = k_j b_j^a(k)^+ = 0, \tag{43a}$$

$$[b_i^a(k), b_j^b(l)] = [b_i^a(k)^+, b_j^b(l)^+] = 0, \tag{43b}$$

$$[b_i^a(k), b_j^b(l)^+] = \frac{L^d}{2|k|}\delta^{ab}\delta_{kl}\left(\delta_{ij} - \frac{k_i k_j}{k^2}\right). \tag{43c}$$

$$H_0 = \frac{1}{L^d}\sum_{k\neq 0} 2k^2 b_j^a(k)^+ b_j^a(k), \qquad (+\text{constant}). \tag{43d}$$

It follows that b and b^+ are energy annihilation and creation operators:

$$[H_0, b_j^a(k)] = -|k| b_j^a(k), \tag{44a}$$

$$[H_0, b_j^a(k)^+] = |k| b_j^a(k)^+. \tag{44b}$$

Starting from the ground state $|0\rangle$ characterised by

$$b_j^a(k)|0\rangle = 0, \tag{45}$$

the Fock space $\mathscr{H}_q$ can now be built up as usual, thus completing the diagonalisation of H_0. Note that the spectrum of H_0 is purely discrete with a gap equal to $2\pi/L$ between the ground state and the first excited state.

Actually, the Fock space construction given above is not the whole story, because the constant field degrees of freedom have not been taken into account. Neither c_k^a nor e_k^a appear in the lowest order hamiltonian H_0. This means that at $g_0 = 0$, the eigenfunctionals of $\hat{H}$ can be written in a factorised form,

$$\hat{\psi}[A] = \varphi(c)\chi[q], \tag{46}$$

where χ is an element of the Fock space $\mathscr{H}_q$ and φ is an arbitrary square integrable function of the constant modes c_k^a. Thus, every eigenstate of H_0 in $\mathscr{H}_q$ gives rise to an infinity of degenerate eigenstates in the full Hilbert space $\hat{\mathscr{H}}$. For example, the ground states of H_0 in $\hat{\mathscr{H}}$ are given by eq. (46), where χ is the wave functional of the Fock space vacuum $|0\rangle$:

$$\chi[q] \propto \exp -\tfrac{1}{2}\int_0^L \mathrm{d}^d x\, q_j^a(x)(-\Delta)^{1/2} q_j^a(x)\,. \tag{47}$$

The infinite degeneracy of the ground state (and of the other states) at $g_0 = 0$ is lifted at first order perturbation theory. As a result, the spectrum of $\hat{H}$ assumes the general form described in sect. 1. In particular, the effective hamiltonian H' gives the level splittings of all those states, which become ground states at $g_0 = 0$.

5. First-order perturbation theory

The first order correction to the leading order hamiltonian H_0 is easily found to be

$$H_1^{(0)} = L^{-d/3}\left\{\tfrac{1}{2}e_k^a e_k^a + \tfrac{1}{4}(f^{abc}c_k^b c_l^c)^2 + c_k^a \int_0^L \mathrm{d}^d x\, f^{abc} q_l^b(x)\partial_k q_l^c(x)\right\}. \tag{48}$$

It depends on the constant modes c_k^a and is therefore capable to lift the degeneracies present at $g_0 = 0$. According to the rules of first order degenerate perturbation theory, $H_1^{(0)}$ must be diagonalised in the subspaces of degenerate states. The first order energy shifts are then equal to the eigenvalues. In particular, the lowest lying energy values are obtained by applying this procedure to the space of ground states as described by eqs. (46) and (47). The relevant matrix elements for this case are

$$(\hat{\psi}_1, H_1^{(0)}\hat{\psi}_2) = L^{-d/3}\int \mathrm{d}c\, \varphi_1(c)^*\{\tfrac{1}{2}e_k^a e_k^a + \tfrac{1}{4}(f^{abc}c_k^b c_l^c)^2\}\varphi_2(c)\,, \tag{49}$$

(the third term in eq. (48) does not contribute, because χ is even under $x \to -x$; the norm of χ is taken equal to 1). It follows that the computation of the first order energy splittings of the lowest lying states exactly amounts to a diagonalisation of the lowest order effective hamiltonian H_0' (cf. sect. 1). Note that the invariance

property (2) required for physical wave functions is a consequence of the corresponding property of reduced wave functionals (cf. discussion after eq. (33)).

I do not know whether the eigenfunctions and eigenvalues of H_0' can be computed analytically. A number of rigorous statements about the spectrum of H_0' can however be made. Most important is the

Theorem: H_0' has purely discrete spectrum, i.e. there is a complete set of normalizable eigenfunctions with every eigenvalue having a finite multiplicity.

At first sight one may not be surprised that the anharmonic oscillator potential

$$V(c) = \tfrac{1}{2}(f^{abc}c_k^b c_l^c)^2 , \tag{50}$$

"confines" the wave functions with finite energy. However, although $V \geqslant 0$ everywhere, $V(c) = 0$ for all abelian constant fields c_k no matter how large. Wave functions could therefore escape along these directions. As is born out in the proof of the theorem given in appendix D, the reason that they do not is that the potential valleys along the abelian c_k's become increasingly narrow as $c_k \to \infty$. There is a close connection here to the discussion in sect. 3. Namely, any abelian c_k can be diagonalised by a constant gauge rotation after which it assumes the toron form (13). The potential valleys discussed here are therefore the constant field views of the toron valleys and the theorem above merely corroborates the conclusion of sect. 3 that the expansion point $A_k = 0$ is stable.

A few properties of the eigenfunction of H_0' can be established rigorously.

(a) Because of elliptic regularity, all eigenfunctions of H_0' are real analytic.

(b) The ground state wave function is not degenerate and can be chosen positive. This follows from a Perron–Frobenius type argument as explained in ref. [8], ch. 8.12, for example. Consequently, the ground state must be invariant under color and space rotations, space reflections and charge conjugations.

(c) From a result of Agmon [9], one deduces that the eigenfunctions of H_0' decay at least as fast as $\exp(-\alpha|c|^{3/2})$ for $c_k \to \infty$, where $\alpha > 0$ is some number independent of the state.

The main conclusion to be drawn from this section is that perturbation theory around the classical vacuum $A_k = 0$ is stabilized at first order. It can now be carried on to any desired order following the well known rules of ordinary degenerate perturbation theory.

6. Degenerate perturbation theory to all orders

This section reviews an economic formulation of degenerate perturbation theory due to Bloch [3]. For detailed derivations and simple examples the reader is referred to the excellent article by Bloch.

Let $\hat{H} = H_0 + \lambda_0 H_1$ be the hamiltonian in the Coulomb gauge (eqs. (40)), but consider λ_0 as an independent coupling constant. Bloch's method yields an expansion in powers of λ_0. In the final expressions one may then set $\lambda_0 = g_0^{2/3}$ and

replace H_1 by the power series (40b). Suppose now that E_0 is an eigenvalue of H_0 and let $\mathscr{H}_0$ be the corresponding eigenspace. The projection operator onto $\mathscr{H}_0$ is denoted by P_0. For small λ_0, one expects that there are eigenstates $|\alpha\rangle$ ($\alpha = 1, 2, 3, \ldots$) of $\hat{H}$ such that

$$\hat{H}|\alpha\rangle = (E_0 + E_\alpha)|\alpha\rangle\,, \qquad E_\alpha = \mathrm{O}(\lambda_0)\,. \tag{51}$$

The linear space spanned by all these states is denoted by $\mathscr{H}$ and the corresponding projector by P. In particular, at $\lambda_0 = 0$ we have $\mathscr{H} = \mathscr{H}_0$ and $P = P_0$.

To first order λ_0, the energy values E_α are equal to the eigenvalues of the operator $\lambda_0 P_0 H_1 P_0$ acting in $\mathscr{H}_0$. Bloch now constructs an operator R acting in $\mathscr{H}_0$, which generalizes $\lambda_0 P_0 H_1 P_0$ in the sense that the eigenvalues of R are equal to the energy values E_α to *all* orders of λ_0. Explicitly, Bloch's formula is

$$R = \lambda_0 P_0 H_1 P_0 + \sum_{\nu=1}^{\infty} \lambda_0^{\nu+1} \sum_{k_1=0}^{\nu} \cdots \sum_{k_\nu=0}^{\nu} \delta_{\nu, k_1+k_2+\cdots+k_\nu}$$
$$\times \chi(k_1, \ldots, k_\nu) P_0 H_1 S^{k_1} H_1 S^{k_2} H_1 \cdots S^{k_\nu} H_1 P_0\,, \tag{52}$$

$$S^0 = -P_0\,, \qquad S^k = \frac{1 - P_0}{(E_0 - H_0)^k}\,, \qquad (k \geqslant 1)\,, \tag{53}$$

$$\chi(k_1, \ldots, k_\nu) = 1\,, \qquad \text{if } \sum_{l=1}^{\mu} k_l \geqslant \mu \qquad \text{for all } \mu = 1, \ldots, \nu\,,$$
$$= 0 \qquad \text{otherwise}\,. \tag{54}$$

In this paper we shall only need the terms up to order λ_0^4. Setting

$$Q_0 = 1 - P_0\,, \qquad a = E_0 - H_0\,, \tag{55}$$

we have

$$R = \sum_{\nu=1}^{\infty} \lambda_0^\nu R_\nu\,, \tag{56a}$$

$$R_1 = P_0 H_1 P_0\,, \tag{56b}$$

$$R_2 = P_0 H_1 \frac{Q_0}{a} H_1 P_0\,, \tag{56c}$$

$$R_3 = P_0 H_1 \frac{Q_0}{a} H_1 \frac{Q_0}{a} H_1 P_0 - P_0 H_1 \frac{Q_0}{a^2} H_1 P_0 H_1 P_0\,, \tag{56d}$$

$$R_4 = P_0 H_1 \frac{Q_0}{a} H_1 \frac{Q_0}{a} H_1 \frac{Q_0}{a} H_1 P_0 - P_0 H_1 \frac{Q_0}{a} H_1 \frac{Q_0}{a^2} H_1 P_0 H_1 P_0$$
$$- P_0 H_1 \frac{Q_0}{a^2} H_1 \frac{Q_0}{a} H_1 P_0 H_1 P_0 - P_0 H_1 \frac{Q_0}{a^2} H_1 P_0 H_1 \frac{Q_0}{a} H_1 P_0$$
$$+ P_0 H_1 \frac{Q_0}{a^3} H_1 P_0 H_1 P_0 H_1 P_0\,. \tag{56e}$$

Although R does not completely solve the problem to compute the energy values E_α, a considerable simplification is achieved, because $\mathscr{H}_0$ usually is a "small" space. In any case, unless $\dim \mathscr{H}_0 = 1$, one will have to diagonalise R by applying perturbation theory again. Before this, one must however take into account that R is in general not hermitian. Rather,

$$BR^+ = RB\,, \qquad B \overset{\text{def}}{=} P_0 P P_0\,, \tag{57}$$

so that a hermitian effective hamiltonian H' can be defined by

$$H' = B^{-1/2} R B^{1/2}\,, \tag{58}$$

H' also acts in $\mathscr{H}_0$ and has the same eigenvalues as R. Noting

$$P = -\sum_{\nu=0}^{\infty} \lambda_0^\nu \sum_{k_1=0}^{\nu} \cdots \sum_{k_{\nu+1}=0}^{\nu} \delta_{\nu,k_1+k_2+\cdots+k_{\nu+1}} S^{k_1} H_1 S^{k_2} H_1 \cdots H_1 S^{k_{\nu+1}}\,, \tag{59}$$

one finds

$$H' = \tfrac{1}{2}(R + R^+) + O(\lambda_0^5)\,. \tag{60}$$

To sum up, the procedure to compute the energy values E_α to order λ_0^4 is to first work out the effective hamiltonian H' according to eqs. (56) and (60) and then to apply perturbation theory to H'. In this paper, only the first of these two steps is completed.

7. One-loop effective hamiltonian

We now apply the method explained in sect. 6 to the ground states of H_0. In this case $E_0 = 0$ and $\mathscr{H}_0$ is the space of wave functions (46) with χ set equal to the Fock space vacuum (47). The elements of $\mathscr{H}_0$ are characterised by square integrable wave functions $\varphi(c)$ and H' is therefore an operator acting on them. This is made explicit by noting that to any order of $g_0^{1/3}$, the perturbation H_1 is a polynomial of c, e, q and p. Because c and e commute with q, p, H_0 and P_0, the expressions (56) for the effective hamiltonian factorise term by term into a Fock space vacuum expectation value and a polynomial of c and e. For example, the leading order contribution to R_2 is (cf. eqs. (48), (56c))

$$\begin{aligned} R_2 &= P_0 H_1^{(0)} \frac{Q_0}{a} H_1^{(0)} P_0 \\ &= \langle 0| \int_0^L \mathrm{d}^d x\, f^{acd} q_i^c \partial_k q_i^d \frac{(1 - |0\rangle\langle 0|)}{-H_0} \int_0^L \mathrm{d}^d y\, f^{bef} q_j^e \partial_k q_j^f |0\rangle \\ &\quad \times L^{-2d/3} c_k^a c_l^b P_0\,. \end{aligned} \tag{61}$$

The action of R_2 on a wave functional $\varphi(c) \cdot \chi[q]$ therefore amounts to multiplying

TABLE 1

List of terms that occur in the expansion of the interaction hamiltonian H_1 in powers of $g_0^{1/3}$ (eq. (40))

$g_0^{0/3}$	$g_0^{1/3}$	$g_0^{2/3}$	$g_0^{3/3}$	$g_0^{4/3}$	$g_0^{5/3}$	$g_0^{6/3}$
e^2		c^2p^2		c^3p^2		e^2q^2
c^4		c^2q^2				c^4p^2
cq^2						
	q^3		cp^2q	$ecpq$	epq^2	ec^2pq
			cq^3	p^2q^2	c^2p^2q	cp^2q^2
				q^4		

Each term is represented symbolically by its operator content disregarding the ordering of the operators. The terms in the lower half of the table do not contribute to the effective hamiltonian H'.

φ with all factors in eq. (61) on the left of P_0. In other words, the contribution to H' (when viewed as an operator acting on φ) is obtained by deleting P_0 on the right-hand side of eq. (61). This rule is valid to all orders and will be applied henceforth without mention.

The aim of this section is to compute H' up to order $g_0^{8/3}$. To this end, one expands H_1 up to order $g_0^{6/3}$, inserts the expansion into eqs. (56) and works out the resulting Fock space vacuum expectation values by Wick's theorem. In the course of this procedure, many terms are generated, which, for symmetry reasons, do not contribute ultimately. It is therefore worthwhile to first survey the terms that can appear. Table 1 lists the possible operator combinations that occur in the expansion of H_1 up to order $g_0^{6/3}$. Not all of these contribute to H', because of the following selection rules:

(a) $\hat{H}$ and $|0\rangle$ are invariant under space reflections. Any term occurring in H' must therefore be even under $c \to -c$, $e \to -e$.

(b) All Fock space vacuum expectation values containing an odd total number of p's and q's vanish.

(c) H' must be real and hermitian. The operator combination ec does therefore not occur.

By inspection, one now easily shows that the terms in the lower half of table 1 contribute at most an additive constant to H' and can therefore be neglected. In other words, to compute H' to order $g_0^{8/3}$, it is sufficient to keep only those terms in the expansion of H_1 with an operator content as listed in the upper half of table 1. This mutilated H_1 is thus given by

$$H_1 = h_1 + h_2 + h_3\,, \tag{62a}$$

$$h_1 = L^{-d/3}\{\tfrac{1}{2}e_k^a e_k^a + \tfrac{1}{4}(f^{abc}c_k^b c_l^c)^2\}\,, \tag{62b}$$

$$h_2 = g_0^{6/3} L^{-4d/3}\tfrac{1}{2}\int_0^L \mathrm{d}^d x\, e_k^a (\mathrm{Ad}\, q_k \Delta^{-1}\, \mathrm{Ad}\, q_l e_l)^a(x)\,, \tag{62c}$$

$$H_0 + g_0^{2/3} h_3 = \tfrac{1}{2} \int_0^L \mathrm{d}^d x \, \{ p_k A_{kl} p_l + q_k B_{kl} q_l \} , \tag{62d}$$

$$A_{kl} = \delta_{kl} + g_0^{4/3} L^{-2d/3} \, \mathrm{Ad}\, c_k \frac{1}{\partial D} \Delta \frac{1}{\partial D} \mathrm{Ad}\, c_l , \tag{63a}$$

$$B_{kl} = -\delta_{kl} D_j D_j + g_0^{4/3} L^{-2d/3} (\mathrm{Ad}\, c_k \, \mathrm{Ad}\, c_l - 2[\mathrm{Ad}\, c_k, \mathrm{Ad}\, c_l]) . \tag{63b}$$

In these expressions, the covariant derivative D_k is defined by

$$D_k = \partial_k + g_0^{2/3} L^{-d/3} \, \mathrm{Ad}\, c_k . \tag{53c}$$

A remarkable feature of eqs. (62) is that h_2 and h_3 are quadratic forms of the "fast" modes p_k and q_k. This structure is typical for one-loop background field calculations.

We now insert eq. (62a) into eqs. (56) and split the computation of $H' = \frac{1}{2}(R + R^+)$ into four steps.

(ii) Terms containing h_1 only. There is actually only one term of this kind, because

$$\frac{Q_0}{a^k} h_1 P_0 = 0 . \tag{64}$$

The contribution to H' is therefore

$$T_1 = (g_0 L^\varepsilon)^{2/3} \frac{1}{L} H_0' . \tag{65}$$

(ii) Terms containing h_2. Since h_2 is of order $g_0^{6/3}$, there is only one contribution here too:

$$T_2 = g_0^{2/3} \langle 0 | h_2 | 0 \rangle .$$

Noting

$$\langle 0 | q_l^a(x) q_j^b(y) | 0 \rangle = \delta^{ab} \frac{1}{L^d} \sum_{k \neq 0} \frac{1}{2|k|} \left(\delta_{lj} - \frac{k_l k_j}{k^2} \right) \mathrm{e}^{\mathrm{i} k \cdot (x-y)} ,$$

one computes

$$T_2 = (g_0 L^\varepsilon)^{8/3} n \frac{d-1}{4d} \left(\frac{1}{L^3} \sum_{k \neq 0} |k|^{-3} \right) \frac{1}{L} e_l^a e_l^a . \tag{66}$$

(iii) Terms containing h_3 only. There are many terms of this kind. However, the calculations can be simplified by noting that the sum of these terms is equal to the ground state energy of the harmonic oscillator (62d), when c_k is considered a fixed background field. Still, a considerable amount of algebra remains (appendix E).

The outcome is

$$T_3 = (g_0 L^\varepsilon)^{4/3} n \frac{(d-1)^2}{4d} \Big(\frac{1}{L} \sum_{k \neq 0} |k|^{-1} \Big) \frac{1}{L} c_l^a c_l^a$$
$$+ (g_0 L^\varepsilon)^{8/3} n \frac{-d^2+7d+18}{96d} \Big(\frac{1}{L^3} \sum_{k \neq 0} |k|^{-3} \Big) \frac{1}{L} (f^{abc} c_l^b c_j^c)^2$$
$$+ (g_0 L^\varepsilon)^{8/3} \frac{d-1}{16d} \Big(\frac{1}{L^3} \sum_{k \neq 0} |k|^{-7} [(6-d)\delta_{lj}\delta_{im} |k|^4 - 5d k_l k_j k_i k_m] \Big)$$
$$\times \frac{1}{L} s^{abcd} c_l^a c_j^b c_i^c c_m^d . \tag{67}$$

(*iv*) *Terms containing* h_1 *and* h_3. Because of eq. (64) such a combination can only occur in R_3 and R_4. It turns out that the rules (a) and (c) above exclude a contribution from R_3 and one is left with

$$T_4 = g_0^{8/3} \tfrac{1}{2} \Big\{ \langle 0 | h_3 \frac{Q_0}{a^3} [h_1, [h_1, h_3]] | 0 \rangle + \langle 0 | [h_1, [h_1, h_3]] \frac{Q_0}{a^3} h_3 | 0 \rangle \Big\} .$$

Noting

$$h_3 = L^{-d/3} c_k^a \int_0^L \mathrm{d}^d x \, f^{abc} q_l^b(x) \partial_k q_l^c(x) + \mathrm{O}(g_0^{2/3}) ,$$

one obtains (after some algebra)

$$T_4 = -(g_0 L^\varepsilon)^{8/3} n \frac{d-1}{16d} \Big(\frac{1}{L^3} \sum_{k \neq 0} |k|^{-3} \Big) \frac{1}{L} (f^{abc} c_l^b c_j^c)^2 . \tag{68}$$

Finally, the contributions T_i, $i = 1, \ldots, 4$, are added up and the following result is obtained:

$$H' = (g_0 L^\varepsilon)^{2/3} \Big[1 + (g_0 L^\varepsilon)^2 n \frac{d-1}{2d} \Big(\frac{1}{L^3} \sum_{k \neq 0} |k|^{-3} \Big) \Big] \frac{1}{2L} e_l^a e_l^a$$
$$+ (g_0 L^\varepsilon)^{2/3} \Big[1 + (g_0 L^\varepsilon)^2 n \frac{-d^2+d+24}{24d} \Big(\frac{1}{L^3} \sum_{k \neq 0} |k|^{-3} \Big) \Big] \frac{1}{4L} (f^{abc} c_l^b c_j^c)^2$$
$$+ (g_0 L^\varepsilon)^{4/3} n \frac{(d-1)^2}{4d} \Big(\frac{1}{L} \sum_{k \neq 0} |k|^{-1} \Big) \frac{1}{L} c_l^a c_l^a$$
$$+ (g_0 L^\varepsilon)^{8/3} \frac{d-1}{16d} \Big(\frac{1}{L^3} \sum_{k \neq 0} |k|^{-7} [(6-d)\delta_{lj}\delta_{im} |k|^4 - 5d k_l k_j k_i k_m] \Big)$$
$$\times \frac{1}{L} s^{abcd} c_l^a c_j^b c_i^c c_m^d . \tag{69}$$

As is seen from this formula, the coefficients of $e_l^a e_l^a$ and $(f^{abc}c_l^b c_j^c)^2$ are not the same. This asymmetry can be removed by rescaling c_k, i.e. by making the substitution*

$$c_k^a \to Z^{1/2} c_k^a, \qquad e_k^a \to Z^{-1/2} e_k^a, \tag{70a}$$

$$Z = 1 + (g_0 L^\varepsilon)^2 n \frac{d^2+11d-36}{72d}\left(\frac{1}{L^3}\sum_{k\neq 0}|k|^{-3}\right) + \cdots . \tag{70b}$$

After that H' assumes the simple form

$$H' = \frac{(g_0 L^\varepsilon)^{2/3}}{L}\{H_0' + (g_0 L^\varepsilon)^{2/3}\alpha_1 c_k^a c_k^a + (g_0 L^\varepsilon)^{6/3}[\alpha_2 H_0' + \alpha_3 s^{abcd} c_k^a c_k^b c_l^c c_l^d + \alpha_4 s^{abcd} c_k^a c_k^b c_k^c c_k^d]\}, \tag{71}$$

$$\alpha_1 = n\frac{(d-1)^2}{4d}\frac{1}{L}\sum_{k\neq 0}|k|^{-1}, \tag{72a}$$

$$\alpha_2 = n\frac{25-d}{72}\frac{1}{L^3}\sum_{k\neq 0}|k|^{-3}, \tag{72b}$$

$$\alpha_3 = \frac{d-1}{16d}\frac{1}{L^3}\sum_{k\neq 0}|k|^{-7}[(6-d)|k|^4 - 15dk_1^2k_2^2], \tag{72c}$$

$$\alpha_4 = -\frac{5(d-1)}{16}\frac{1}{L^3}\sum_{k\neq 0}|k|^{-7}(k_1^4 - 3k_1^2k_2^2). \tag{72d}$$

To go from here to the effective hamiltonian as quoted in sect. 1, it remains to evaluate the one-loop momentum sums in the coefficients α_i. Following the rules of dimensional regularisation**, the sums are first rewritten with the help of the heat kernel on S^1:

$$f(t) \overset{\text{def}}{=} \sum_{\nu=-\infty}^{\infty} \exp\{-t(2\pi\nu)^2\}, \tag{73}$$

$$\frac{1}{L^s}\sum_{k\neq 0}|k|^{-s} = \frac{1}{\Gamma(\frac{1}{2}s)}\int_0^\infty \mathrm{d}t\, t^{\frac{1}{2}s-1}(f^d - 1). \tag{74}$$

The coefficients α_i then become

$$\alpha_1 = n\frac{(d-1)^2}{4d\sqrt{\pi}}\int_0^\infty \mathrm{d}t\, t^{-1/2}(f^d - 1), \tag{75a}$$

$$\alpha_2 = n\frac{25-d}{36\sqrt{\pi}}\int_0^\infty \mathrm{d}t\, t^{1/2}(f^d - 1), \tag{75b}$$

* Only the spectrum of H' is observable; this is not affected by a change of variables.

** For more details about dimensional regularisation with compact dimensions see e.g. ref. [10].

$$\alpha_3 = \frac{d-1}{30\sqrt{\pi}} \int_0^\infty \mathrm{d}t\, t^{5/2}[(6-d)f^{d-1}f'' - (d^2-7d+21)f^{d-2}(f')^2], \tag{75c}$$

$$\alpha_4 = -\frac{d-1}{6\sqrt{\pi}} \int_0^\infty \mathrm{d}t\, t^{5/2}[f^{d-1}f'' - 3f^{d-2}(f')^2]. \tag{75d}$$

Some properties of the heat kernel on S^1 are listed in appendix B (in the notation used there $f(t) = F_t(0;1)$). It follows from these that the coefficients α_i are meromorphic functions of d, the representations (75) being valid for low Re d. Only α_2 has a pole at $d=3$:

$$\alpha_2 = \frac{11n}{9(4\pi)^2}\frac{1}{\varepsilon} + \mathrm{O}(1). \tag{76}$$

This pole is exactly cancelled by the coupling constant renormalization (11). In other words, H' as given by eq. (71) is finite at $d=3$, provided only the coupling constant is renormalized in the usual way. It is not difficult to compute the finite part of α_2 and the values of the other α's at $d=3$ (see appendix F for a sample calculation). With this last step completed, the effective hamiltonian as presented in sect. 1 is obtained.

We conclude this section by remarking that the effective hamiltonian for the $2+1$ dimensional theory can be quickly obtained from the above formulae simply by evaluating the α's at $d=2$ (there are no poles there and the coupling constant need not be renormalized). At $d=1$ the expected null result is obtained.

8. Concluding remarks

The calculations in this paper rely on the Schrödinger representation, i.e. on the use of wave functionals and of the Hamilton operator. The Schrödinger representation of the ϕ^4 theory was recently investigated by Symanzik [11]. He found that, as expected, all amplitudes were well defined in the dimensionally regularised theory and that, furthermore, the wave functionals could be renormalized so as to make them finite and non-trivial in the cutoff free theory. To the extent that no unexpected ultraviolet divergencies were encountered, the results obtained here show that the same is true in the gauge theory case, too. It is possible to compute the effective hamiltonian H' by a purely euclidean method. Namely, one first calculates the euclidean transition amplitude to go from a constant gauge field to another in perturbation theory and expands for large times T. H' can then be read off essentially from the term proportional to T. Using the Lorentz gauge, I computed the one-loop coefficients α_i along these lines and reproduced eqs. (75). One may therefore be confident that the hamiltonian method not only yields finite but also correct results. From the calculational point of view, the euclidean and the hamiltonian technique are about equally voluminous. However, most of the steps necessary to compute the effective hamiltonian are programmable (the expansion

of $\hat{H}$ and Bloch's formulae, for example) so that one may very well be able to go beyond the one-loop level.

A striking result of the present investigation is the exact degeneracy of the central sectors to all orders of perturbation theory. This is a truly non-linear effect, which does not take place in the abelian gauge theory, where the sectors are split by energy gaps proportional to g^2/L. From lattice gauge theories (or the "electric flux" interpretation [5]) one expects, on the other hand, that the gaps in the non-abelian case *increase* with L. While such a behaviour would be difficult to obtain in perturbation theory, a non-perturbative contribution always is accompanied by a power L^p, where p can be any number greater than -1. For this reason, the perturbative degeneracy of the central sectors is welcome and one may even hope to estimate the string tension by computing the non-perturbative splitting of the sectors semi-classically.

Many of the technical difficulties that arise when perturbation theory is applied to gauge fields on a torus are due to the existence of torons. I therefore first considered to calculate the energy spectrum for gauge fields confined to a sphere S^3, which is simply connected and does not allow for toron solutions. However, this possibility must be dismissed for another reason: S^3 is a curved space so that relative to the infinite volume theory the wave functions are distorted *locally*. The approach of the finite volume masses to the infinite volume values is therefore slow (i.e. like $1/L$ instead of $\exp(-mL)$) and there would be little hope to extract accurate numbers for the mass ratios from perturbation theory (see ref. [1] for further discussion).

Practically all results obtained in this paper carry over to gauge theories on a finite periodic lattice. In particular, central sectors, torons and the effective hamiltonian H' exist as before. To lowest order, H' is still equal to H'_0, but the one-loop coefficients a_i now depend on L in a way, which is specific to the particular lattice action chosen.

Appendix A

SU(n) NOTATIONS

The Lie algebra $\mathfrak{su}(n)$ of SU(n) consists of all complex $n \times n$ matrices X with

$$X^+ = -X, \qquad \mathrm{Tr}\, X = 0\,. \tag{A.1}$$

Let T^a, $a = 1, \ldots, n^2-1$, be a basis of such matrices satisfying

$$\mathrm{Tr}\,(T^a T^b) = -\tfrac{1}{2}\delta^{ab}\,. \tag{A.2}$$

The structure constants f^{abc} and the totally symmetric tensor d^{abc} are then defined by

$$[T^a, T^b] = f^{abc} T^c\,, \tag{A.3}$$

$$\{T^a, T^b\} = -\frac{1}{n}\delta^{ab} + id^{abc}T^c . \tag{A.4}$$

Both tensors are real and

$$f^{ade}f^{bde} = n\delta^{ab}, \tag{A.5}$$

$$d^{ade}d^{bde} = \frac{1}{n}(n^2-4)\delta^{ab} . \tag{A.6}$$

The totally symmetric tensor s^{abcd}, which occurs in the effective hamiltonian (eq. (3e)), is given by

$$\begin{aligned} s^{abcd} = {} & \tfrac{1}{12}n(d^{abe}d^{cde} + d^{ace}d^{bde} + d^{ade}d^{bce}) \\ & + \tfrac{2}{3}(\delta^{ab}\delta^{cd} + \delta^{ac}\delta^{bd} + \delta^{ad}\delta^{bc}) . \end{aligned} \tag{A.7}$$

Its group theoretical meaning is explained below. For any $X \in \mathfrak{su}(n)$ define a linear mapping

$$\begin{aligned} &\mathrm{Ad}\, X: \qquad \mathfrak{su}(n) \to \mathfrak{su}(n) , \\ &\mathrm{Ad}\, X(Y) = [X, Y] \qquad \text{for all } Y \in \mathfrak{su}(n) . \end{aligned} \tag{A.8}$$

With respect to the basis T^a, $\mathrm{Ad}\, X$ is represented by a matrix $(\mathrm{Ad}\, X)^{ab}$ so that

$$\mathrm{Ad}\, X(T^b) = T^a(\mathrm{Ad}\, X)^{ab} . \tag{A.9}$$

Explicitly, writing

$$X = X^a T^a , \tag{A.10}$$

we have

$$(\mathrm{Ad}\, X)^{ab} = f^{acb}X^c , \tag{A.11}$$

$$(\mathrm{Ad}\, X(Y))^a = (\mathrm{Ad}\, X)^{ab}Y^b . \tag{A.12}$$

The matrices $(\mathrm{Ad}\, X)^{ab}$ form the adjoint representation of $\mathfrak{su}(n)$. In particular,

$$[\mathrm{Ad}\, X, \mathrm{Ad}\, Y] = \mathrm{Ad}\,[X, Y] , \tag{A.13}$$

$$\mathrm{Tr}\,(\mathrm{Ad}\, X\ \mathrm{Ad}\, Y) = 2n\ \mathrm{Tr}\,(XY) . \tag{A.14}$$

The meaning of the symbol s^{abcd} is now made clear by the following

Lemma: Let $X_1, \ldots, X_4$ be four elements of $\mathfrak{su}(n)$. Then

$$\frac{1}{4!}\sum_\sigma \mathrm{Tr}\,(\mathrm{Ad}\, X_{\sigma(1)}\, \mathrm{Ad}\, X_{\sigma(2)}\, \mathrm{Ad}\, X_{\sigma(3)}\, \mathrm{Ad}\, X_{\sigma(4)}) = s^{abcd}X_1^a X_2^b X_3^c X_4^d , \tag{A.15}$$

where σ runs over all permutations of (1, 2, 3, 4).

Proof: For any choice of real numbers λ_i set

$$X = \sum_{i=1}^{4} \lambda_i X_i .$$

The left- and right-hand sides of eq. (A.15) can then be written as

$$\text{lhs} = \frac{1}{4!} \cdot \frac{\partial^4}{\partial\lambda_1 \partial\lambda_2 \partial\lambda_3 \partial\lambda_4} \text{Tr}\,(\text{Ad}\,X)^4\,,$$

$$\text{rhs} = \frac{1}{4!} \cdot \frac{\partial^4}{\partial\lambda_1 \partial\lambda_2 \partial\lambda_3 \partial\lambda_4} s^{abcd} X^a X^b X^c X^d\,.$$

It is therefore sufficient to show that

$$\text{Tr}\,(\text{Ad}\,X)^4 = s^{abcd} X^a X^b X^c X^d\,,$$

for all $X \in \mathfrak{su}(n)$. To this end, first note that

$$\begin{aligned}\text{Tr}\,(\exp \text{Ad}\,X) &= -2\,\text{Tr}\,(T^a\, \mathrm{e}^X T^a\, \mathrm{e}^{-X}) \\ &= (\text{Tr}\,\mathrm{e}^X)(\text{Tr}\,\mathrm{e}^{-X}) - 1\,.\end{aligned}$$

Identifying the terms of order X^4 we have

$$\text{Tr}\,(\text{Ad}\,X)^4 = 2n\,\text{Tr}\,X^4 + 6(\text{Tr}\,X^2)^2\,.$$

Finally, using eq. (A.2) and

$$X^2 = -\frac{1}{2n} X^a X^a + \tfrac{1}{2} i d^{abc} X^a X^b T^c\,,$$

one obtains

$$\begin{aligned}\text{Tr}\,(\text{Ad}\,X)^4 &= \tfrac{1}{4} n d^{abe} d^{cde} X^a X^b X^c X^d + 2(X^a X^a)^2 \\ &= s^{abcd} X^a X^b X^c X^d\,,\end{aligned}$$

as required. □

Appendix B

HEAT KERNEL ON S^1

The heat kernel on S^1 is defined by ($t > 0$, $z \in \mathbb{C}$)

$$F_t(z; L) = \frac{1}{L} \sum_{\nu=-\infty}^{\infty} \exp\left\{-t\left(\frac{2\pi\nu}{L}\right)^2 + i\frac{2\pi\nu}{L} z\right\}. \tag{B.1}$$

It is the fundamental solution of the heat equation on S^1:

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial z^2}\right) F_t(z; L) = 0\,, \tag{B.2}$$

$$F_t(z + L; L) = F_t(z; L)\,, \tag{B.3}$$

$$\lim_{t\to 0} F_t(z; L) = \sum_{\nu=-\infty}^{\infty} \delta(z - \nu L)\,, \qquad (z \in \mathbb{R})\,, \tag{B.4}$$

Using the Poisson summation formula one obtains

$$F_t(z;L) = (4\pi t)^{-1/2} \sum_{\nu=-\infty}^{\infty} \exp\left\{-\frac{1}{4t}(z-\nu L)^2\right\}. \tag{B.5}$$

For real z, $F_t(z;L)$ is hence real and positive. Furthermore

$$F_t(0;L) = (4\pi t)^{-1/2}\left\{1+O\left(\exp\left[-\frac{L^2}{4t}\right]\right)\right\}, \qquad (t\to 0), \tag{B.6}$$

$$F_t(0;L) = \frac{1}{L}\left\{1+O\left(\exp\left[-t\left(\frac{2\pi}{L}\right)^2\right]\right)\right\}, \qquad (t\to\infty). \tag{B.7}$$

Appendix C

COMPUTATION OF $\mathscr{E}_0[\varphi]$

We first determine the eigenvalues of $\Omega[\varphi]$. This is easy, because the toron field $A_k[\varphi]$ is independent of x. The eigenfunctions of $\Omega[\varphi]$ are therefore plane waves:

$$q_j^a(x) = v_j^a \, e^{ik\cdot x}, \tag{C.1}$$

$$k_j = \frac{2\pi}{L}\nu_j, \qquad \nu_j \in \mathbb{Z}, \tag{C.2}$$

The eigenvalue equation and the gauge condition (23b) then become algebraic:

$$(k_l - i \operatorname{Ad} A_l[\varphi])^2 v_j = E v_j, \tag{C.3}$$

$$(k_l - i \operatorname{Ad} A_l[\varphi]) v_l = 0. \tag{C.4}$$

The conditions (23a) requires that the three matrices v_l have zeros on their diagonal if $k = 0$. The matrices $\operatorname{Ad} A_l[\varphi]$ commute and can be simultaneously diagonalised. This solves eq. (C.3). The constraint (C.4) merely reduces the multiplicities of the energy levels by 1*. The result is:

$$E_0(k) = k^2, \qquad k \neq 0, \qquad \text{multiplicity } 2(n-1), \tag{C.5}$$

$$E_{\alpha\beta}(k) = \left(k_l + \frac{1}{L}(\varphi_l^\alpha - \varphi_l^\beta)\right)^2, \qquad \alpha \neq \beta, \qquad \text{multiplicity } 2. \tag{C.6}$$

We now compute $\mathscr{E}_0[\varphi]$ using Pauli–Villars regulators. The reason for not using dimensional regularisation is the latent danger inherent to this method that φ-dependent terms, which are more than logarithmically divergent, are regularized to zero. Thus, let M_j and ε_j $(j = 1, \ldots, \nu)$ be regulator masses and alternating signs such that

$$\sum_{j=1}^{\nu} \varepsilon_j = -1, \qquad \sum_{j=1}^{\nu} \varepsilon_j M_j^{2p} = 0, \qquad (p = 1, 2, \ldots, \nu-1). \tag{C.7}$$

* Strictly speaking this is only true if $E > 0$. However, zero modes occur only for torons $A_k[\varphi]$ at the boundary of the toron manifold and even in that case it is not necessary to count them correctly, because they do not contribute to $\mathscr{E}_0[\varphi]$.

Then the regularized frequency sum is

$$\mathscr{E}_0[\varphi] = \sum_{\alpha\neq\beta}\sum_{k}\{\sqrt{E_{\alpha\beta}(k)} + \sum_{j=1}^{\nu}\varepsilon_j\sqrt{E_{\alpha\beta}(k)+M_j^2}\}\,. \tag{C.8}$$

Here and below all terms not depending on φ_l^α are dropped. Noting

$$\sqrt{E} + \sum_{j=1}^{\nu}\varepsilon_j\sqrt{E+M_j^2} = \frac{1}{\Gamma(-\frac{1}{2})}\int_0^\infty \mathrm{d}t\, t^{-3/2}\Big(1+\sum_{j=1}^{\nu}\varepsilon_j\,\mathrm{e}^{-tM_j^2}\Big)\,\mathrm{e}^{-tE}\,, \tag{C.9}$$

$$\sum_k \mathrm{e}^{-tE_{\alpha\beta}(k)} = \prod_{l=1}^{3}\Big(\sum_{\nu_l=-\infty}^{\infty}\exp\Big\{-\frac{t}{L^2}(2\pi\nu_l+\varphi_l^\alpha-\varphi_l^\beta)^2\Big\}\Big)\,, \tag{C.10}$$

we have

$$\mathscr{E}_0[\varphi] = \sum_{\alpha\neq\beta}\frac{1}{\Gamma(-\frac{1}{2})}\int_0^\infty \mathrm{d}t\, t^{-3/2}\Big(1+\sum_{j=1}^{\nu}\varepsilon_j\,\mathrm{e}^{-tM_j^2}\Big)$$
$$\times\prod_{l=1}^{3}\Big(\sum_{\nu_l=-\infty}^{\infty}\exp\Big\{-\frac{t}{L^2}(2\pi\nu_l+\varphi_l^\alpha-\varphi_l^\beta)^2\Big\}\Big)\,. \tag{C.11}$$

Next, the sums are converted using Poisson's summation formula (cp. appendix B):

$$\sum_{\nu_l=-\infty}^{\infty}\exp\Big\{-\frac{t}{L^2}(2\pi\nu_l+\varphi_l^\alpha-\varphi_l^\beta)^2\Big\}$$
$$=\frac{L}{\sqrt{4\pi t}}\sum_{\nu_l=-\infty}^{\infty}\exp\Big\{-\frac{1}{4t}(\nu_l L)^2+i\nu_l(\varphi_l^\alpha-\varphi_l^\beta)\Big\}\,. \tag{C.12}$$

When inserted into eq. (C.11) one then sees that only the term with $\nu_1=\nu_2=\nu_3=0$ is ultra-violet divergent. But this term is also independent of φ_l^α so that the interesting φ-dependent part of $\mathscr{E}_0[\varphi]$ is convergent for $M_j\to\infty$. Furthermore, the t-integral becomes elementary in this limit and the final result is

$$\mathscr{E}_0[\varphi] = -\frac{1}{\pi^2 L}\sum_{\alpha\neq\beta}\sum_{\nu\neq 0}(\nu^2)^{-2}\cos\nu\cdot(\varphi^\alpha-\varphi^\beta)\,, \tag{C.13}$$

(the minus sign stems from $\Gamma(-\frac{1}{2}) = -\sqrt{4\pi}$; ν denotes the integer vector (ν_1,ν_2,ν_3) and $\varphi^\alpha = (\varphi_1^\alpha,\varphi_2^\alpha,\varphi_3^\alpha)$).

Appendix D

SPECTRUM OF H_0'

The proof of the theorem in sect. 5 is based on a criterion valid for general Schrödinger operators

$$A = -\Delta + V, \tag{D.1}$$

where $V \geqslant 0$ is a continuous function on $\mathbb{R}^m$. Under these conditions, A is essentially self-adjoint on $C_0^\infty(\mathbb{R}^m)$, the space of all infinitely differentiable wave functions $\psi(x)$ with compact support (ref. [8], theorem 10.28). This means that A extends from $C_0^\infty(\mathbb{R}^m)$ to a unique self-adjoint operator, which will also be denoted by A.

Criterion: Suppose there is a continuous function $\lambda(x)(x \in \mathbb{R}^m)$ such that $\lambda \geqslant 0$ and

$$(\psi, A\psi) \geqslant \int \mathrm{d}^m x\, \lambda(x) |\psi(x)|^2 , \qquad \text{for all } \psi \in C_0^\infty(\mathbb{R}^m) . \tag{D.2}$$

Suppose furthermore that $\lambda(x) \to \infty$ as $x \to \infty$. Then, A has purely discrete spectrum.

This criterion follows easily from a formula for the bottom of the essential spectrum of A quoted by Agmon [9]. Alternatively, one may adapt the proof of theorem 13.16 of ref. [8], which treats the case where the potential V itself goes to infinity in all directions (i.e. in this case one may choose $\lambda = V$).

It remains to find a function λ for H_0'. To this end, note that the potential V (eq. 50) can be written as

$$V(c) = c_1^a M^{ab} c_1^b + \tfrac{1}{2} \sum_{k,j=2}^{d} (f^{abc} c_k^b c_j^c)^2 , \tag{D.3}$$

$$M^{ab} = \sum_{k=2}^{d} (f^{ace} c_k^c)(f^{bde} c_k^d) , \tag{D.4}$$

It follows that for any $\psi \in C_0^\infty$

$$(\psi, H_0'\psi) \geqslant \tfrac{1}{2} \int \mathrm{d}c\, \psi(c)^* \left(-\frac{\partial^2}{\partial c_1^{a2}} + c_1^a M^{ab} c_1^b \right) \psi(c) . \tag{D.5}$$

Now, M is symmetric with eigenvalues $\omega_a^2 \geqslant 0$ $(a = 1, \ldots, n^2 - 1)$. Furthermore, M is independent of c_1 so that at fixed c_k, $k \geqslant 2$, the operator on the right-hand side of eq. (D.5) is just a harmonic oscillator hamiltonian with frequencies ω_a. Performing the c_1 integrations first, it follows that*

$$(\psi, H_0'\psi) \geqslant \tfrac{1}{2} \int \mathrm{d}c \left(\sum_a \omega_a \right) |\psi(c)|^2 . \tag{D.6}$$

Noting

$$\sum_a \omega_a \geqslant \left(\sum_a \omega_a^2 \right)^{1/2} = (\mathrm{Tr}\, M)^{1/2} = \left(n \sum_{k=2}^{d} c_k^a c_k^a \right)^{1/2} , \tag{D.7}$$

one thus obtains

$$(\psi, H_0'\psi) \geqslant \tfrac{1}{2}\sqrt{n} \int \mathrm{d}c \left(\sum_{k=2}^{d} c_k^a c_k^a \right)^{1/2} |\psi(c)|^2 . \tag{D.8}$$

* This is the crucial step in the proof. It realizes the intuitive insight that wave functions in narrow potential valleys have high energy, because the kinetic term forces them to spread into regions where the potential is not small.

Finally, the estimations are repeated with c_1 replaced by c_2 etc. and the resulting inequalities (D.8) are then averaged. This yields the symmetric bound

$$(\psi, H_0'\psi) \geqslant \frac{1}{2d}\sqrt{n(d-1)}\int \mathrm{d}c\,(c_k^a c_k^a)^{1/2}|\psi(c)|^2\,. \tag{D.9}$$

Hence

$$\lambda(c) = \frac{1}{d}\sqrt{n(d-1)}(c_k^a c_k^a)^{1/2}\,, \tag{D.10}$$

has all the properties necessary for the criterion to apply.

Appendix E

COMPUTATION OF T_3

The ground state energy of the harmonic oscillator (62d) is given by

$$T_3 = \tfrac{1}{2}\,\mathrm{Tr}'\,\{[(PAP)^{1/2}B(PAP)^{1/2}]^{1/2}\}\,. \tag{E.1}$$

Here, P denotes the projector

$$P_{kl} = \delta_{kl} - \partial_k \Delta^{-1}\partial_l\,, \tag{E.2}$$

and $\mathrm{Tr}'\,\{\cdots\}$ means the trace over the space of periodic functions $f_k^a(x)$ with

$$\int_0^L \mathrm{d}^d x\, f_k^a(x) = 0\,. \tag{E.3}$$

In a condensed notation, we have from eqs. (63)

$$(PAP)^{1/2} = P + \tfrac{1}{2}(g_0 L^\varepsilon)^{4/3}\frac{1}{L^2}P\cdot \mathrm{Ad}\,c\,\frac{1}{\partial D}\Delta\frac{1}{\partial D}\,\mathrm{Ad}\,c\cdot P$$

$$-\tfrac{1}{8}(g_0 L^\varepsilon)^{8/3}\frac{1}{L^4}\Big(P\cdot \mathrm{Ad}\,c\,\frac{1}{\partial D}\Delta\frac{1}{\partial D}\,\mathrm{Ad}\,c\cdot P\Big)^2 + \cdots\,, \tag{E.4}$$

$$PBP = P(-\Delta)P - (g_0 L^\varepsilon)^{2/3}\frac{2}{L}P\,\mathrm{Ad}\,c\cdot\partial P$$

$$-(g_0 L^\varepsilon)^{4/3}\frac{1}{L^2}(P\,\mathrm{Ad}\,c\cdot\mathrm{Ad}\,cP - P\cdot\mathrm{Ad}\,c\,\mathrm{Ad}\,c\cdot P + 2P\cdot[\mathrm{Ad}\,c, \mathrm{Ad}\,c]\cdot P)\,.$$

Furthermore, using

$$\frac{1}{\partial D}\Delta\frac{1}{\partial D} = \Delta^{-1} - (g_0 L^\varepsilon)^{2/3}\frac{2}{L}\Delta^{-1}\,\mathrm{Ad}\,c\cdot\partial\Delta^{-1}$$

$$+(g_0 L^\varepsilon)^{4/3}\frac{3}{L^2}\Delta^{-1}\,\mathrm{Ad}\,c\cdot\partial\Delta^{-1}\,\mathrm{Ad}\,c\cdot\partial\Delta^{-1} + \cdots\,, \tag{E.6}$$

one first obtains $(PAP)^{1/2}B(PAP)^{1/2}$ and then the square root of this operator as a power series of $g_0^{2/3}$. A typical term in the latter expansion is

$$X = (-\Delta)^{-3/2} P \,\mathrm{Ad}\, c \cdot \partial \,\mathrm{Ad}\, c \cdot \partial P \,. \tag{E.7}$$

The trace of such operators is easily worked out in momentum space:

$$\begin{aligned} \mathrm{Tr}'\, X &= -(d-1) \sum_{k \neq 0} |k|^{-3} k_l k_j \,\mathrm{Tr}\,(\mathrm{Ad}\, c_l \,\mathrm{Ad}\, c_j) \\ &= -\frac{d-1}{d} \sum_{k \neq 0} |k|^{-1} \,\mathrm{Tr}\,(\mathrm{Ad}\, c_l \,\mathrm{Ad}\, c_l) \,. \end{aligned} \tag{E.8}$$

Here, the trace on the right-hand side refers to color only. Up to an additive constant, the outcome then is

$$\begin{aligned} T_3 = &-(g_0 L^{\varepsilon})^{4/3} \frac{1}{L^2} \frac{(d-1)^2}{4d} \sum_{k \neq 0} |k|^{-1} \,\mathrm{Tr}\,(\mathrm{Ad}\, c_l \,\mathrm{Ad}\, c_l) \\ &-(g_0 L^{\varepsilon})^{8/3} \frac{1}{L^4} \Big\{ \frac{1}{4d} \sum_{k \neq 0} |k|^{-3} \,\mathrm{Tr}\,([\mathrm{Ad}\, c_l, \mathrm{Ad}\, c_j][\mathrm{Ad}\, c_l, \mathrm{Ad}\, c_j]) \\ &+\frac{(d-1)(d-6)}{16d} \sum_{k \neq 0} |k|^{-3} \,\mathrm{Tr}\,(\mathrm{Ad}\, c_l \,\mathrm{Ad}\, c_l \,\mathrm{Ad}\, c_j \,\mathrm{Ad}\, c_j) \\ &+\frac{5(d-1)}{16} \sum_{k \neq 0} |k|^{-7} k_l k_j k_i k_m \,\mathrm{Tr}\,(\mathrm{Ad}\, c_l \,\mathrm{Ad}\, c_j \,\mathrm{Ad}\, c_i \,\mathrm{Ad}\, c_m) \Big\} \,. \end{aligned} \tag{E.9}$$

Finally, using some identities from appendix A, one finds

$$\mathrm{Tr}\,(\mathrm{Ad}\, c_l \,\mathrm{Ad}\, c_l) = -n c_l^a c_l^a \,, \tag{E.10}$$

$$\begin{aligned} &\mathrm{Tr}\,(\mathrm{Ad}\, c_l \,\mathrm{Ad}\, c_l \,\mathrm{Ad}\, c_j \,\mathrm{Ad}\, c_j) \\ &= s^{abcd} c_l^a c_l^b c_j^c c_j^d - \tfrac{1}{6} \mathrm{Tr}\,([\mathrm{Ad}\, c_l, \mathrm{Ad}\, c_j][\mathrm{Ad}\, c_l, \mathrm{Ad}\, c_j]) \,, \end{aligned} \tag{E.11}$$

$$\mathrm{Tr}\,([\mathrm{Ad}\, c_l, \mathrm{Ad}\, c_j][\mathrm{Ad}\, c_l, \mathrm{Ad}\, c_j]) = -n (f^{abc} c_l^b c_j^c)^2 \,, \tag{E.12}$$

$$\mathrm{Tr}\,(k \cdot \mathrm{Ad}\, c)^4 = k_l k_j k_i k_m s^{abcd} c_l^a c_j^b c_i^c c_m^d \,. \tag{E.13}$$

When these relations are inserted into eq. (E.9), one obtains the result quoted in sect. 7.

Appendix F

EVALUATION OF α_1 AT $d = 3$

Define

$$h(t) = f\Big(\frac{t}{4\pi}\Big) = \sum_{\nu=-\infty}^{\infty} \exp\,(-t\pi\nu^2) \,. \tag{F.1}$$

The coefficient α_1 is then given by (cf. eq. (75))

$$\alpha_1 = n\frac{(d-1)^2}{8\pi d}\int_0^\infty \mathrm{d}t\, t^{-1/2}(h^d-1)\,, \qquad \mathrm{Re}\, d<1\,. \tag{F.2}$$

Using the duality relation (B.5),

$$h\left(\frac{1}{t}\right) = \sqrt{t}\,h(t)\,, \tag{F.3}$$

we have

$$\begin{aligned}\int_0^1 \mathrm{d}t\, t^{-1/2}(h^d-1) &= -2+\int_1^\infty \mathrm{d}t\, t^{(d-3)/2}h^d \\ &= -\frac{2d}{d-1}+\int_1^\infty \mathrm{d}t\, t^{(d-3)/2}(h^d-1)\,.\end{aligned}$$

It follows that

$$\alpha_1 = n\frac{(d-1)^2}{8\pi d}\left\{-\frac{2d}{d-1}+\int_1^\infty \mathrm{d}t(t^{-1/2}+t^{(d-3)/2})(h^d-1)\right\}. \tag{F.4}$$

This representation is valid for all d. In particular,

$$\alpha_1|_{d=3} = \frac{n}{6\pi}\left\{-3+\int_1^\infty \mathrm{d}t(t^{-1/2}+1)(h^3-1)\right\}. \tag{F.5}$$

Finally, substituting the series (F.1) for h and performing the integral numerically, one gets

$$\alpha_1|_{d=3} = -\frac{n}{4\pi}1.89153165\ldots. \tag{F.6}$$

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